1

Units and Dimensions and Mach Numbers

About twenty to twenty-five thousand years ago, an enormous meteor hit the earth in northern Arizona, approximately sixty kilometers southeast of the present-day city of Flagstaff. This meteor, composed mostly of iron, had a diameter of about 40 meters and a mass of around 263,000 metric tons. Its impact velocity was approximately 72,000 kilometers per hour or 20,000 meters per second. With this information, it is easy to determine that the kinetic energy of the meteor at the instant of collision was $e = (1/2)mU^2 = 5.26 \times 10^{16}$ joules. This is about 625 times more than the energy released by an ordinary atomic bomb.

This immense meteor struck the earth with such enormous force that it dug a crater 1,250 meters in diameter and 170 meters deep. More than 250 million metric tons of rock and dirt were displaced. The sound created by the impact must have been totally awesome.

As we shall see shortly, the velocity of sound in air is given by the equation $C = 20.07\sqrt{T}$, where C is the sonic velocity in meters per second and T is the absolute temperature of the air in degrees kelvin. For example, suppose that the air temperature is 68°F (fahrenheit) = 20°C (celsius) = 20°C + 273 = 293°K (kelvin); then the velocity of sound is $C = 20.07\sqrt{293} = 344$ m/s.

Had there been a city of Flagstaff when the meteor hit the earth, the people living there—60 kilometers away—would have heard the noise of the impact about 175 seconds after it occurred. Had there been a Los Angeles—620 kilometers to the west—the sound waves created by the collision would have reached there about 30 minutes later.

Over the years, scientists and engineers have devised several "numbers" that they use in mathematical analyses and computations involving the motion of objects moving through fluids such as water and air. By far the best known of these important numbers is the *Mach number*. It is highly likely, for example, that just about everyone has heard that the Concorde supersonic airliner, at cruising speed, has a Mach number of 2.0.

The Mach number, Ma, is defined as the velocity of an object moving through a fluid (e.g., water or air) divided by the velocity of sound in the same fluid. That is, $Ma = U/C$. In our meteor collision problem, $U = 20{,}000$ m/s and $C = 344$ m/s. Consequently, $Ma = 20{,}000/344 = 58$. This is a very large Mach number. The meteor was moving so fast just prior to impact that it created temperatures sufficiently high to ionize the air completely. This means that the molecules and atoms composing the air—mostly nitrogen and oxygen—were disintegrated into a gas called a "plasma." Ordinarily, even in high-speed aerodynamics, Mach numbers are much lower than the Mach number associated with the Arizona meteor. Typically, they are less than about 10. Never mind. For the moment, we simply want to present a definition of this quantity called the Mach number.

Units and Dimensions

In all fields of science and engineering, the subject of *units* and *dimensions* plays a very important role. In the physical and mathematical analyses of these fields, it is necessary to specify the fundamental dimensions of a measurement system and to define precisely the basic units to be used.

We should be careful to distinguish between the two quantities: units and dimensions. For example, *length* is a fundamental dimension; its units of measurement may be in feet, in miles, or in kilometers. *Time* is another fundamental dimension; its units may be expressed in seconds, in weeks, or in years; and so on.

For our purpose, there are five fundamental dimensions. These are the following:

mass, M, or force, F
length, L
time, T
electric current, A
temperature, θ

Sometimes it is preferable to use the dimension force, F, instead of mass, M. The two are easily interchanged because from Newton's equation, force = mass × acceleration, $F = M \times L/T^2$.

In our analysis, we consider the following systems of units:

the International System (SI) or metric system
the English or engineering system

For each of these, table 1.1 lists the proper units for the corresponding *fundamental dimensions*. For example, the SI or metric column indicates that the newton is the unit of force, the kilogram is the unit of mass, and the meter is the unit of length. Also, for each of the systems, the dimensions and units of several *derived quantities* are shown.

International System (SI) or Metric System

The metric system of units was originated in France following the French Revolution in the late eighteenth century. Being based on the units of meters, kilograms, and seconds, the metric system was referred to as the MKS system for many years. In 1960, it was replaced by what is called the International System (SI), which has been adopted by nearly all nations; eventually it will be used throughout the world.

Many scientists continue to use the centimeter-gram-second (CGS) system of units. This is the same as the SI system except

TABLE 1.1

Systems of units and corresponding dimensions

Quantities and dimensions	*SI or metric*	*English or engineering*
Fundamental		
Force F	Newton	Pound
Mass M	Kilogram	Slug
Length L	Meter	Foot
Time T	Second	Second
Current A	Ampere	Ampere
Temperature θ absolute	Kelvin, °K	Rankine, °R
Temperature θ relative	Celsius, °C	Fahrenheit, °F
Derived		
Velocity U L/T	Meters/second	Feet/second
Pressure p F/L^2	Newtons/meter^2	Pounds/foot^2
Density ρ M/L^3	Kilograms/meter^3	Slugs/foot^3
Energy e FL	Newton meters or joules	Foot pounds
Power P FL/T	Joules per second or watts	Foot/pounds/second

that the centimeter replaces the meter as the unit of length and the gram replaces the kilogram as the unit of mass.

A word about the temperature units indicated in table 1.1. In the SI system, absolute and relative temperatures are related by the equation °K = °C + 273.2. In addition, we have the relationship °C = (5/9)(°F − 32), where °F is degrees fahrenheit.

English or Engineering System

Most of the countries of the world have now adopted the SI system of units. Only in the United States, Great Britain, and some other English-speaking countries is the English/engineering system still being used. However, it is slowly being replaced by the much simpler and more logical SI system.

As table 1.1 indicates, the *pound* and the *slug* are the customary units for force (F) and mass (M). However, in Great Britain, the *poundal* is frequently taken as the unit of *force* (F). In this case, the unit of mass (M) is the pound.

In the English/engineering system, absolute and relative temperatures are related by the equation °R = °F + 459.7. In addition, we have °F = (9/5)°C + 32, where °C is degrees celsius.

Conversion of Units and Some Examples

A short list of numerical conversion factors is presented in table 1.2. Much longer lists are presented in many references. For example, a long table of conversion factors is given in Lide (1994).

TABLE 1.2

A short list of conversion factors between English or engineering and International System or metric

1 inch = 2.540 centimeters	1 pound = 0.4536 kilograms
1 foot = 30.48 centimeters	1 pound = 4.448 newtons
1 meter = 3.281 feet	1 slug = 32.2 pounds
1 mile = 5280 feet	1 kilogram = 2.205 pounds
1 mile = 1.609 kilometers	1 kilogram = 9.82 newtons
1 nautical mile = 6076.4 feet	1 calorie = 4.186 joules
1 nautical mile = 1852 meters	1 horsepower = 0.7457 kilowatts
°F = (9/5)°C + 32	°C = (5/9)(°F − 32)

PROBLEM 1. In the SI system of units, the acceleration due to gravity is $g = 9.82$ m / s^2. What is its value in the English / engineering system?

$$g = 9.82 \frac{\text{m}}{\text{s}^2} = 9.82 \frac{(3.281 \text{ ft})}{\text{s}^2} = 32.2 \frac{\text{ft}}{\text{s}^2}.$$

PROBLEM 2. In the SI system of units, the density of air is $\rho = 1.20$ kg / m^3. What is its value in the English / engineering system?

$$\rho = 1.20 \frac{\text{kg}}{\text{m}^3} = 1.20 \frac{(2.205 \text{ lb})}{(3.281 \text{ ft})^3} = 0.0749 \frac{\text{lb}}{\text{ft}^3},$$

$$\rho = 0.0749 \frac{\left(\frac{1}{32.2} \text{ slug}\right)}{\text{ft}^3} = 0.00233 \frac{\text{slug}}{\text{ft}^3}.$$

PROBLEM 3. In the English / engineering system, the wind pressure on a tall building is $p = 45$ lb / ft^2. What is its value in the SI system?

$$p = 45 \frac{\text{lb}}{\text{ft}^2} = 45 \frac{(4.448 \text{ newton})}{\left(\frac{1}{3.281} \text{ meter}\right)^2},$$

$$p = 45(4.448)(3.281)^2 \frac{\text{newton}}{\text{meter}^2},$$

$$p = 2{,}155 \frac{\text{N}}{\text{m}^2} = 2{,}155 \text{ pascal}.$$

Prefixes for SI Units

These days we hear a lot about nanoseconds, megawatts, kilograms, and micrometers. We note that each of these SI units has a prefix. These prefixes give the precise size of the unit. A list of these prefixes, and their symbols and sizes, is given in table 1.3.

Dimensional Analysis

A topic closely related to the subject of units and dimensions, indeed one which is built entirely on the concept and theory of dimensions, is *dimensional analysis*. It is extremely important in

TABLE 1.3

Prefixes for SI units

Prefix	*Symbol*	*Size: 10^k Value of k*	*Prefix*	*Symbol*	*Size: 10^k Value of k*
deka	da	1	deci	d	−1
hecto	h	2	centi	c	−2
kilo	k	3	milli	m	−3
mega	M	6	micro	μ	−6
giga	G	9	nano	n	−9
tera	T	12	pico	p	−12
peta	P	15	femto	f	−15
exa	E	18	atto	a	−18

many areas of science and engineering, especially in the subjects of fluid mechanics and aerodynamics. We will not go much beyond a brief introduction to the topic. Numerous references are available: Barenblatt (1996), Ipsen (1960), and Langhaar (1951).

An Example: Flight of a Baseball and the Reynolds Number

To illustrate how dimensional analysis is used, we analyze a problem that is well known to nearly everyone: the flight of a baseball. In this case, a sphere of diameter D moves through a fluid (i.e., air) with velocity U. The fluid has density ρ_a and viscosity μ. The stitches and seam on the baseball create a rough surface that, like sandpaper, can be described by a certain roughness height ϵ.

The resistance force, F, that the fluid exerts on the sphere depends on a number of things. Mathematically, this dependence can be expressed in the following way:

$$F = f(D, U, \rho_a, \mu, \epsilon). \tag{1.1}$$

This relationship says that the resistance force F depends on—or, as a mathematician would say, is a function of—the diameter, D, and velocity, U, of the sphere, the density, ρ_a, and viscosity, μ, of the fluid through which the sphere is moving, and the roughness of the sphere, ϵ.

Altogether there are *six* variables in our problem; these are listed in equation (1.1). Collectively, these variables possess *three* of the fundamental dimensions: mass (M), length (L), and time (T). So the values of two important quantities in our dimensional analysis problem are $m = 6$ (number of physical variables) and $n = 3$ (number of fundamental dimensions).

The basic principle of dimensional analysis is contained in the following statement: Consider a system in which there are m independent dimensional variables that affect the system. Furthermore, there are n fundamental dimensions among these m quantities. Then it is possible to construct $(m - n)$ dimensionless parameters to relate these quantities functionally.

On this basis, in our problem, with $m = 6$ and $n = 3$, we can expect to construct $(m - n) = 6 - 3 = 3$ dimensionless parameters. Sure enough, if we were to go through the details of the entire dimensional analysis, we would obtain the following expression:

$$\frac{F}{\frac{1}{2}\rho_a AU^2} = f\left(\frac{\rho_a UD}{\mu}, \frac{\epsilon}{D}\right). \tag{1.2}$$

The quantity on the left-hand side of this equation expresses the resistance force; it is a dimensionless quantity. Likewise, the two quantities within the brackets on the right-hand side are also dimensionless quantities. Incidentally, when we say "dimensionless," we simply mean that the exponents of each of the fundamental dimensions in a particular parameter add up to zero. For practice, try checking the dimensions of the parameters of equation (1.2).

Equation (1.2) indicates that the term for the resistance force, $F(1/2)\rho_a AU^2$, is a function of the two quantities $\rho_a UD/\mu$ and

ϵ/D. We can rewrite this expression in the following way:

$$F = \tfrac{1}{2}\rho_a C_D A U^2, \qquad (1.3)$$

in which $A = (\pi/4)D^2$ is the projected or shadow area of the sphere and C_D is the *drag coefficient*. It is clear that

$$C_D = f(Re, \epsilon/D), \qquad (1.4)$$

where $Re = \rho_a UD/\mu$ is a quantity called the *Reynolds number*; the parameter ϵ/D is termed the relative roughness. In words, equation (1.4) says that the drag coefficient, C_D, depends on—or is a function of—the Reynolds number, Re, and the relative roughness, ϵ/D.

If the sphere is completely smooth—like a ping-pong ball, for example—then the roughness $\epsilon = 0$. In this case, the drag coefficient depends only on the Reynolds number. That is,

$$C_D = f(Re). \qquad (1.5)$$

We note that the Reynolds number, $Re = \rho_a UD/\mu$, contains the viscosity, μ. Consequently, this important dimensionless number gives a measure of the importance of viscosity in a particular fluid flow phenomenon.

In later chapters, where we deal with baseballs, golf balls, and other objects moving through air, we shall take a close look at drag coefficients, Reynolds numbers, and the roughness caused by baseball seams and golf ball dimples. Why do we want to know about these things? Well, quite likely one of the main reasons is to be able to compute the trajectories—the flight paths—of baseballs and golf balls as they sail through the sky, in which case, as we shall see later on, it is absolutely imperative to have quantitative information about drag coefficients, lift coefficients, and the like.

However, our interest may go far beyond the task of simply calculating sporting ball flight paths. The same mathematics and physics are involved—though generally somewhat more complicated—if we want to compute the trajectories of projectiles, missiles, rockets, and yes, even ski jumpers.

Velocity of Sound in a Gas

When a sound wave passes through a gas—for example, air—the gas is slightly compressed momentarily by the wave. If we were to carry out a detailed analysis of this event, we would make the basic assumption that there is no gain or loss of heat into or out of the gas. In terms of thermodynamics, this says that the process is *adiabatic*. Utilizing this assumption and employing the so-called general gas law, we obtain the equation

$$C = \sqrt{\frac{\gamma R_*}{m} T}, \tag{1.6}$$

in which C is the velocity of sound in the gas, γ is the specific heat ratio of the gas, R_* is the universal gas constant, m is the molecular weight of the gas, and T is the absolute temperature. For air, $\gamma = 1.405$, $R_* = 8.314$ joules/°K mol, and $m = 29 \times 10^{-3}$ kg/mol. With these values, equation (1.6) becomes

$$C = 20.07\sqrt{T}, \tag{1.7}$$

which is the equation for the velocity of sound in air. It is interesting to note that the sonic velocity depends only on the temperature. For example, if $T = 20°C = 293°K$, then equation (1.7) gives $C = 344$ m/s, a result we obtained earlier in the chapter.

Velocity of Sound in a Liquid

Although it is usually assumed that liquids, including water, are incompressible, it turns out that they are, in fact, slightly compressible. If K is the so-called coefficient of compressibility of a liquid and ρ is its density, it can be shown that

$$C = \frac{1}{\sqrt{\rho K}}. \tag{1.8}$$

This is the equation for the velocity of sound in a liquid. For example, the value of K for sea water at 20°C is $K = 4.25 \times 10^{-10}$ m^2/newton and the density is $\rho = 1{,}025$ kg/m^3. If these numbers are substituted into equation (1.8), we obtain $C = 1{,}515$ m/s as the velocity of sound in sea water. At this same temperature, the velocity of sound in air is $C = 344$ m/s. Thus, the sonic velocity in the ocean is more than four times larger than it is in air. Most likely, whales and dolphins have known for quite a long time that vocal transmissions are much swifter *below* the surface.

External Forces in Fluid Flow Phenomena

We have seen that when the force of viscosity is the most important external force in a fluid flow, then dimensional analysis indicates that the Reynolds number, *Re*, is the important parameter involved in the problem. Likewise, if the force of compressibility is predominant, then a similar analysis predicts that the Mach number, *Ma*, is the crucial parameter of the phenomenon.
In the same way, if gravity is the major external force, then dimensional analysis would indicate that the dimensionless number called the *Froude number*, $Fr = U/\sqrt{gD}$, is the important parameter. Finally, if the major external force is due to surface tension, σ, then the *Weber number*, $We = \rho U^2 D/\sigma$ is the critical flow parameter. These are the important dimensionless numbers we mentioned at the beginning of the chapter.

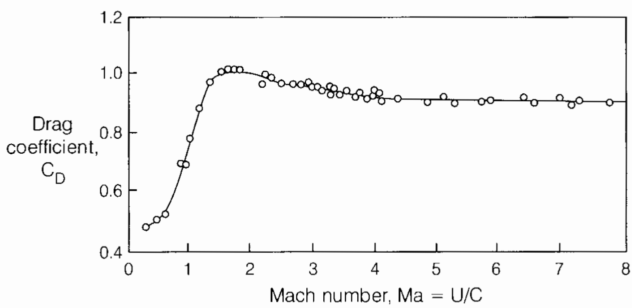

FIG. 1.1

Drag coefficient, C_D, versus Mach number, *Ma*, for smooth spheres. (From Barenblatt 1996.)

An Example: Flight of a Supersonic Sphere and the Mach Number

Suppose that a smooth sphere of diameter D moves at a very high velocity U through a compressible gas—for example, air. It is assumed that the effects of viscosity can be neglected.

In this case, the drag coefficient, C_D, depends only on the Mach number, $Ma = U/C$. This result, predicted by dimensional analysis, is confirmed by experimental results. A plot is presented in figure 1.1 of the drag coefficient versus the Mach number for the flow of air past a smooth sphere.

In the figure we note the following:

1. For values of Ma less than about 0.5 (i.e., in the subsonic region), the value of C_D has approximately the same value, $C_D = 0.48$, as in incompressible flow (e.g., the flight of a baseball or a golf ball).
2. For values of Ma from 0.5 to 1.5 (i.e., in the transonic region), there is a sharp increase in the value of the drag coefficient.
3. For a value of Ma equal to about 1.5 (i.e., in the supersonic region), the drag coefficient reaches a maximum value, $C_D = 1.02$.
4. For values of Ma greater than approximately 3.0 (i.e., in the supersonic-hypersonic region), the drag coefficient has a constant value of about 0.90.

The most important characteristic of supersonic flow—that is, a flow with Mach number larger than 1.0—is the appearance of shock waves. A reference that deals with this and numerous other topics of aerodynamics is Anderson (1991).

In conclusion, it should be mentioned that the Mach number is named after the noted Austrian physicist Ernst Mach (1838–1916). During the latter part of the nineteenth century, Mach carried out many studies on mechanics and aerodynamics, including extensive work involving wind tunnel experimentation. In addition to his numerous contributions in various areas of physical science, Mach is well known for his accomplishments in the fields of psychology and philosophy.