Figure 24.4. A plane function invariant under the octahedral group $\mathcal{O}$. Centers of the 6 faces are colored black. The 8 vertices are gray dots, marking centers of 3-fold rotational symmetry. The centers of edges—centers of half-turns—appear between two bits of cloud.

Astute readers might connect the formula in (24.2) to (24.1) and realize that it amounts to requiring that $n - m \equiv 0 \pmod 4$ for nonzero coefficients. This is actually how the image was created. Equation (24.2) gave us another chance to think about the technique of coset averaging.

The Icosahedral Group. Felix Klein wrote an entire book about $\mathcal{I}$. It is a glorious but painful object. This group can be obtained from the tetrahedral group by introducing a single 5-fold rotation with an obscure axis: the vector $(\phi, 1, 0)$, where $\phi = (1 + \sqrt{5})/2$ is the golden ratio. (Remember how this curious number popped up in our study of 5-fold symmetry in Chapter 13? It's back!) To get the full group, we must find all compositions of this rotation with elements of $\mathcal{T}$. There are 60 of them.

The details will be handled in an exercise. Here, I'll just report that this transfers to the plane as (drumroll, please)

$$\gamma_5(z) = \frac{\phi(1 - i\phi)z + 1 + 2i}{\sqrt{5}z + \phi(1 - i\phi)}.$$

I never would have guessed.

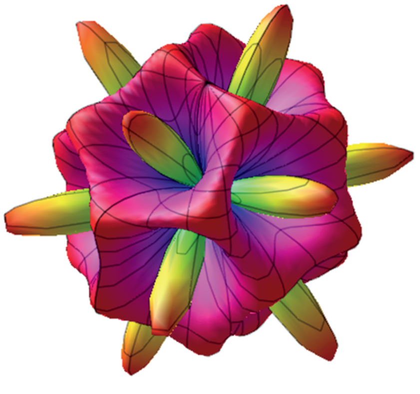

Figure 24.5. A sphere deformed by a function with icosahedral symmetry.

To make functions invariant under $\mathcal{I}$, we could simply average over all 60 elements. I did this to produce a surface in space, analogous to the one with tetrahedral symmetry in Figure 24.2 (right). Figure 24.5 shows how much fun this can be. But let's move on.

Let's use coset averaging to make icosahedral functions in the plane. We start with functions invariant under $\mathcal{T}$ and average over the five cosets of the