Three Classes of Time-Delay Systems for which the Aizerman Conjecture is True

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***Abstract*—This short note addresses the Aizerman problem for retarded second-order systems with multiple delays and first-order neutral systems with a single delay. It is proved that the Aizerman conjecture is true for second-order systems with delays not involving derivatives. For systems with multiple delays involving the first derivative, a delay-dependent class of systems is identified for which the conjecture is true. For first-order neutral systems, the conjecture is true, provided that an additional delay-independent requirement is met. The proof is based on the Popov absolute stability criterion.**

***Index Terms*—Absolute stability, Aizerman problem, time-delay systems.**

I. Introduction

In the earlier paper [1] we considered the Aizerman problem for the following time-delay system:

$$\ddot{x}(t)+a_1\dot{x}(t)+\sum_{j=1}^{m} b_j x(t-\tau_j)+\varphi(x)=0\,. \tag{1}$$

The function $\varphi(x)$ is assumed to satisfy the sector condition:

$$0\le \frac{\varphi(x)}{x}<\mu\,. \tag{2}$$

The problem can be described as follows. Consider, along with (1), the linear case, i.e., $\varphi(x)=ax$. Aizerman conjecture states: if the zero solution of the system is globally asymptotically stable (GAS) in the

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linear case, it is also GAS for all functions $\varphi(x)$, satisfying (2) with $\mu = a$.

The matter is well settled for systems without delays: the Aizerman conjecture is true for second-order systems and, generally, false for systems of order three and higher [2]. For time-delay systems the problem is still open [3], except that Rasvan himself proved that the conjecture is true for retarded first-order systems [4] and the earlier contribution [1] identified two types of second-order systems for which the conjecture is true, namely, retarded systems with a single delay and a delay-dependent class of systems with multiple delays not involving derivatives. This paper is the continuation of [1], in which it was proved that the Aizerman conjecture is true for systems described by (1) if

$$a_1 > \sum_{j=1}^{m} |b_j| \tau_j \tag{3}$$

The result to be proved in Section II is that the Aizerman conjecture is true for (1), regardless of the magnitudes of delays. In Section III, a result similar to the one in [1], will be proved for the case when delays involve the first derivative of the unknown function. Finally, in Section IV, we prove the conjecture for a broad delay-independent class of first-order neutral systems.

Similarly to [1], we use the frequency-domain approach. To this end, we define the transfer function of the linear terms. For (1) it has the form:

$$W(s) = \left[s^2 + a_1 s + \sum_{j=1}^{m} b_j e^{-\tau_j s} \right]^{-1}. \tag{4}$$

In proving the results of the paper, we shall rely on the Popov criterion: the zero solution of the nonlinear system is GAS if there exists a constant β, such that for all values of ω, including infinity, the following inequality holds:

$$\mu^{-1} + \mathrm{Re}\left[(1 + i\omega\beta) W(i\omega) \right] > 0. \tag{5}$$

In all three cases considered, application of this criterion will require some additional algebraic manipulations.

II. Proof for Delays not Involving Derivatives

In order to prove the Aizerman conjecture for (1), we must also consider the linear case:

$$\ddot{x}(t) + a_1\dot{x}(t) + ax + \sum_{j=1}^{m} b_j x(t-\tau_j) = 0 . \tag{6}$$

It is an immediate consequence of the result from [5] that the zero solution of (6) is GAS if and only if both of the following inequalities hold:

$$\sum_{j=1}^{m}\left|b_j\right| < a ,\ 2\sum_{j=1}^{m}\left|b_j\right| < a_1^2 ,\ a_1 > 0 . \tag{7}$$

The first of these inequalities implies that in order for the zero solution of (1) to be GAS, the nonlinearity $\varphi(x)$ must satisfy the inequality

$$\varphi(x) > x\sum_{j=1}^{m}\left|b_j\right| . \tag{8}$$

This suggests defining a new nonlinearity:

$$f(x) = \varphi(x) - x\sum_{j=1}^{m}\left|b_j\right| , \tag{9}$$

which clearly satisfies (2) as long as (8) is satisfied and $\varphi(x) < \mu x$.

The transfer function of the linear part then becomes:

$$W(s) = \left[s^2 + a_1 s + \sum_{j=1}^{m}\left|b_j\right| + \sum_{j=1}^{m} b_j e^{-\tau_j s} \right]^{-1} . \tag{10}$$

Expansion of the Popov inequality (5) yields:

$$\sum_{j=1}^{m}\left|b_j\right| + \left(a_1\beta - 1\right)\omega^2 - \beta\omega\sum_{j=1}^{m} b_j \sin\omega\tau_j + \sum_{j=1}^{m} b_j \cos\omega\tau_j \ge 0 . \tag{11}$$

We can replace this inequality with

$$\sum_{j=1}^{m}\left[\left|b_j\right| + \gamma_j\left(a_1\beta - 1\right)\omega^2 - \beta\omega b_j \sin\omega\tau_j + b_j\cos\omega\tau_j\right] \ge 0 , \tag{12}$$

where γ_j, j=1…m is a set of positive real numbers such that $\sum_{j=1}^{m}\gamma_j = 1$. Clearly, (12) holds if each summand is nonnegative. Using a well known trigonometric identity, we can rewrite this in the form:

$$\left|b_j\right| + \gamma_j\left(a_1\beta - 1\right)\omega^2 + \sqrt{b_j^2 + \left(\beta\omega b_j\right)^2}\,\sin(\omega\tau_j + \phi) \ge 0\,. \tag{13}$$

It is easy to demonstrate that (13) holds if β is chosen to satisfy the double inequality:

$$\frac{\gamma_j a_1}{\left|b_j\right|} - \sqrt{\frac{\gamma_j\left(\gamma_j a_1^2 - 2\left|b_j\right|\right)}{\left|b_j\right|^2}} < \beta < \frac{\gamma_j a_1}{\left|b_j\right|} + \sqrt{\frac{\gamma_j a_1^2 - 2\left|b_j\right|}{\left|b_j\right|^2}}\,. \tag{14}$$

Furthermore, we set:

$$\gamma_j = \left|b_j\right| \Big/ \sum\nolimits_{j=1}^m \left|b_j\right|\,. \tag{15}$$

The inequality (14) then becomes:

$$\frac{a_1}{\sum_{j=1}^m \left|b_j\right|} - \sqrt{\frac{a_1^2 - 2\sum_{j=1}^m \left|b_j\right|}{\left(\sum_{j=1}^m \left|b_j\right|\right)^2}} < \beta < \frac{a_1}{\sum_{j=1}^m \left|b_j\right|} + \sqrt{\frac{a_1^2 - 2\sum_{j=1}^m \left|b_j\right|}{\left(\sum_{j=1}^m \left|b_j\right|\right)^2}} \tag{16}$$

In light of the second of the inequalities (7), the radicands are positive. Therefore, stability of the zero solution of the linear equation (6) implies that a constant β can be found to satisfy the Popov inequality, which means that zero solution of (1) is GAS. The Aizerman conjecture for this type of systems is proved.

III. Extension to the Case of Time Delays Involving First Derivatives

Consider the system:

$$\ddot{x}(t) + a_1\dot{x}(t) + \sum_{j=1}^m b_{1j}\dot{x}(t-\tau_j) + \sum_{j=1}^m b_{0j}x(t-\tau_j) + \varphi(x) = 0\,. \tag{17}$$

The corresponding linear system is

$$\ddot{x}(t) + a_1\dot{x}(t) + ax + \sum_{j=1}^m b_{1j}\dot{x}(t-\tau_j) + \sum_{j=1}^m b_{0j}x(t-\tau_j) = 0\,. \tag{18}$$

The first of the inequalities (7) is still a necessary condition for stability of the zero solution of (18), except that b_j is replaced with b_{0j}. For this reason, we once again define the new nonlinearity by (9). The transfer function of the linear part becomes:

$$W(s)=\left[s^2+a_1s+\sum_{j=1}^{m}\left|b_{0j}\right|+\sum_{j=1}^{m}\left(b_{0j}+b_{1j}s\right)e^{-\tau_j s}\right]^{-1}. \tag{19}$$

Expansion of the Popov inequality (5) yields:

$$\sum_{j=1}^{m}\left|b_{0j}\right|+\left(a_1\beta-1\right)\omega^2+\omega\sum_{j=1}^{m}\left(b_{1j}-b_{0j}\beta\right)\sin\omega\tau_j+\sum_{j=1}^{m}\left(b_{0j}+b_{1j}\beta\omega^2\right)\cos\omega\tau_j\geq 0. \tag{20}$$

The method from the previous Section does not produce a satisfactory result in this case. However, it is still possible to use the approach from Section V of [1] and use the estimate $\chi\sin\alpha\chi\leq\alpha\chi^2$, valid for $\alpha>0$. Because of this estimate, we can state that (20) holds for all real values of ω if there exists $\beta>0$, such that the following inequality holds for all real values of ω:

$$\sum_{j=1}^{m}\left|b_{0j}\right|+\sum_{j=1}^{m}b_{0j}\cos\omega\tau_j+\left(a_1\beta-1\right)\omega^2-\omega^2\sum_{j=1}^{m}\left|b_{1j}-b_{0j}\beta\right|\tau_j-\omega^2\sum_{j=1}^{m}\left|b_{1j}\beta\right|\geq 0. \tag{21}$$

This can be assured by choosing $\beta>0$ to satisfy:

$$\left(a_1-\sum_{j=1}^{m}\left|b_{0j}\right|\tau_j-\sum_{j=1}^{m}\left|b_{1j}\right|\right)\beta\geq 1+\sum_{j=1}^{m}\left|b_{1j}\right|. \tag{22}$$

Clearly, this can be done if and only if

$$a_1>\sum_{j=1}^{m}\left|b_{0j}\right|\tau_j+\sum_{j=1}^{m}\left|b_{1j}\right|. \tag{23}$$

We have thus identified a delay-dependent class of systems, for which the Aizerman conjecture is true. It is worth noting that if delays do not involve derivatives, (23) reduces to (3).

IV. Neutral First-Order Systems with a Single Delay

Consider now the first-order neutral system:

$$\dot{x}(t)+b_1\dot{x}(t-\tau)+bx(t-\tau)+\varphi(x)=0. \tag{24}$$

The corresponding linear system is:

$$\dot{x}(t)+ax(t)+b_1\dot{x}(t-\tau)+bx(t-\tau)=0 \tag{25}$$

The necessary conditions for stability of the zero solution of the linear system are [5]:

$$|b| < a \,,\ |b_1| < 1 \tag{26}$$

Therefore, as in the previous Sections, we define the new nonlinearity:

$$f(x) = \varphi(x) - |b|\,x \,. \tag{27}$$

The transfer function of the linear part becomes:

$$W(s) = \left[s + |b| + \left(s + b_1 \right) e^{-\tau s} \right]^{-1} . \tag{28}$$

Expansion of the Popov inequality yields:

$$|b| + \beta\omega^2 + \left(b + b_1\beta\omega^2 \right)\cos\omega\tau + \left(b_1 - b\beta \right)\omega\sin\omega\tau \ge 0 \,. \tag{29}$$

As in Section II, we can rewrite (29) using a well-known trigonometric identity:

$$|b| + \beta\omega^2 + \sqrt{\left(b + b_1\beta\omega^2 \right)^2 + \left[\left(b_1 - b\beta \right)\omega \right]^2}\,\sin\left(\omega\tau + \phi \right) \ge 0 \tag{30}$$

This inequality holds for all real values of ω if

$$\left(1 - b_1^2 \right)\omega^2 - \left(b_1^2 - 2|b|\beta + b^2\beta^2 \right) \ge 0 \tag{31}$$

holds for all real values of ω.

Due to the second of the inequalities (26), the coefficient at ω^2 in (31) is positive. Therefore, (31) holds for all real values of ω if the constant β is chosen to satisfy

$$\frac{|b|\left(1 - \sqrt{1 - b_1^2} \right)}{b^2} < \beta < \frac{|b|\left(1 + \sqrt{1 - b_1^2} \right)}{b^2} \,. \tag{32}$$

Note that the radicands are positive due to the second of the inequalities (26). This completes the proof of the conjecture in this case, except that the method fails if $b = 0$. Therefore, for first-order neutral systems that conjecture is true if $b \neq 0$.

V. Conclusion

The results of this note, together with those of [1], [3], and [4], show that the Aizerman conjecture is true for all first- and second-order systems with delays not involving derivatives as well as for second-order systems with a single delay. For second-order systems with multiple delays involving first derivatives, a delay-dependent class of systems, for which the conjecture is true, is identified. The latter result relies on a very coarse estimate, and it is possible that future research will lead to its refinement.

Section IV is a first step in addressing the problem for neutral systems in that the conjecture is proved for a delay-independent class of first-order neutral systems. For second-order neutral systems, the problem is still open. It is worth noting that the Popov criterion fails for such systems. While this fact is certainly not a refutation of the conjecture, it does suggest that for second-order neutral systems, the answer to the Aizerman problem may be in the negative.

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