Ptolemy's Map Projections and Coordinate Lists

In introducing the principles of map-making, in *Geography* 1.20, Ptolemy refers to the two kinds of map, spherical and plane, and points out that although maps on spheres keep the earth's spherical shape and consequently preserve perfectly the relative proportions of intervals on the earth, they are usually too small to show all the things one wants to map, and they cannot be surveyed by the eye in a single glance. Plane maps, on the other hand, although they fulfill the two last demands, require "some method" to satisfy the first two.

Plane maps do not have to represent the spatial relationships between places in a definite, quantitative way. We know that world maps in classical antiquity could be highly schematic, like the circular map that appears in a group of medieval Greek astronomical manuscripts, in which, for example, Egypt and the upper Nile are portrayed as an oblique rectangle straddling a horizontal chord representing the Tropic of Cancer.[42] Herodotus (4.36, Loeb 2:235) and Aristotle (*Meteor.* 2.5 362b12, Loeb 181–183) both describe the world maps (*periodoi*) of their time as circular, with a ring-shaped Ocean entirely surrounding the land-mass of the *oikoumenē*; though the fact that both authors describe the plan of these maps as laughable shows that they had a conception that a map should somehow portray the regions of the world with roughly correct relative positions and sizes. The *Tabula Peutingeriana* ("Peutinger Table"), a medieval Latin map of the Roman Empire and its environs that is an indirect copy of a lost fourth century map, illustrates still another possibility: the map is a rectangular strip, nearly 7 meters wide but only 34 centimeters high, so that north-south distances are greatly compressed relative to east-west distances, and all outlines are accordingly distorted.[43] The lost source-map was designed primarily to exhibit the network of roads with their distances, for which there was no need to preserve much semblance to the shapes on the globe, and the map's dimensions were likely dictated by the original medium, possibly a papyrus roll.[44]

On the other hand, any world map that displayed localities in relation to a "graticule" (grid of principal parallels and meridians) would be practically forced to conform to a *projection*, that is, a mathematically definable rule for establishing a unique point on the planar surface corresponding to each point determined by a given parallel and meridian on the globe. And there is considerable evidence, especially in Strabo, that the "revision" of the traditional map of the world that Eratosthenes (c. 285–194 B.C.) presented in the third book of his

[42]Neugebauer 1975b (with illustration).

[43]A color reproduction of a section of the *Tabula Peutingeriana* is shown in *History of Cartography*, vol. 1, plate 5; for the whole, see Miller 1916 or Bosio 1983.

[44]Dilke 1985, 113–120.

Geography extensively employed geometrical constructions in relation to a grid of parallels and meridians.[45]

We may presume, therefore, that the history of map projections began not later than Eratosthenes in the third century B.C. Ptolemy tells us that Marinos criticized "absolutely all" previous methods of making plane maps, which implies that there had in fact been considerable experimentation with making such maps prior to his time. We know almost nothing about what these methods were, with the exception of the geographer Strabo's verbal description of a graticule suitable for the world map (early first century A.D.). Although frustratingly lacking in technical detail, the passage is worth quoting as the only surviving discussion of the topic before Ptolemy:[46]

> But [a world map] requires a large globe, so that the aforesaid segment of it [containing the *oikoumenē*], being such a small fraction of it, will be sufficient to hold the suitable parts of the *oikoumenē* with clarity and give an appropriate display to the spectators. Now if one can fashion [a globe] this large, it is better to do it in this way; and let it have a diameter not less than ten feet. But if one cannot make [a globe] of this size or not much smaller, one ought to draw [the map] on a planar surface of at least seven feet. For it will make little difference if instead of the circles, i.e., the parallels and meridians with which we show the *klimata* and directions and other variations and placements of the parts of the earth relative to each other and to the heavens, we draw straight lines, with parallel lines for the parallels, and perpendicular lines for the [meridians] perpendicular to them. [This is permissible] because the intellect is able easily to transfer the shape and size seen by the sight on a planar surface to the [imagined] curved and spherical [surface]. The same will apply to oblique circles [on the globe] and straight lines [corresponding to them on the map]. And though it is true that the meridians everywhere, since they are all described through the pole, all converge to one point on the globe, nevertheless it will not matter if on the planar surface one makes the straight lines for the meridians bend together only a little. For even this is not necessary in many situations when the lines [representing the meridians and parallels on the globe] are transferred to the planar surface and drawn as straight lines, nor is the convergence [of the meridians] as conspicuous as the curvature [of the globe].

Strabo evidently has in mind two ways of drawing the lines representing the circles of latitude and longitude. In the first, parallels of latitude are represented by horizontal straight lines, and meridians by vertical straight lines, so

[45] Strabo 2.1 (Loeb 1:253–361).
[46] Strabo 2.5.10 (Loeb 1:449–451).

that every parallel intersects every meridian exactly at right angles and the meridians, being represented by parallel lines, do not converge at all toward the north. In the second, the parallels of latitude are again drawn as horizontal lines, but the meridians converge a little at the north end of the map. This might mean that the meridians are drawn as straight lines inclined slightly from the vertical as if to meet at a point somewhere above the north end of the map, in which case they cannot all be perpendicular to the parallels. But Strabo may merely have in mind a slight inward curvature of the meridians only at the very top of the map, as if to suggest schematically their ultimate convergence while keeping them otherwise perpendicular to the equator and parallel to each other.[47]

Strabo appraises these representations only from the point of view of their adequacy in giving the general visual impression of the *oikoumenē* as it would be seen on a globe, and so he says nothing about what metrical properties of the map on the globe, such as distance, area, or direction, are preserved in either planar projection. At least in the version of the map with the meridians drawn throughout as parallels, one would presumably have kept the horizontal intervals between meridians on the map in correct proportion to the longitudinal intervals between the actual meridians, and likewise the vertical intervals between parallels on the map proportional to the latitudinal intervals between the actual parallels. Strabo's first grid would therefore have resulted, in modern terminology, in an *equirectangular cylindrical* projection, in which distances measured along all meridians would be portrayed in correct ratio to distances measured along, at most one parallel north of the equator, or along the equator itself. If he intended the meridians in the second version of the grid to be convergent straight lines, the projection would have resembled the so-called *Donis* or *trapezoidal* projection invented by Nicolaus Germanus in the 1460s, in which the meridians are drawn as straight lines converging so that distances measured along the top and bottom parallels are in correct ratio to each other, and only the central meridian and central parallel are at right angles and in correct proportionality of distances to each other.[48]

Whatever the variety of projections Marinos had to choose from, he had, according to Ptolemy, adopted just that mapping which was least successful in preserving proportionality of distances. In Marinos' map graticule (Fig. 10), the parallels of latitude are represented by a set of parallel straight lines and the meridians by another set of parallel straight lines at right angles to them, as in Strabo's projection. But Marinos also specified how distances along the paral-

[47]This is definitely what Strabo has in mind when he applies the same vocabulary to the courses of the Rhine and the Pyrenees in 4.5.1 (Loeb 2.253).

[48]"Donis" is a misreading of *donnus* or *dominus*, prefixed as an honorific to Nicolaus' name, apparently because he was a Benedictine.

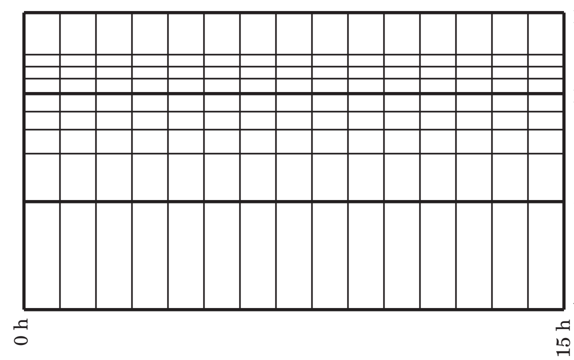

Fig. 10. Graticule of Marinos' projection

lels and meridians were to be represented in the projection. The ratio of the spacing of the lines separated by a given number of degrees of latitude relative to that of the lines separated by the same number of degrees of longitude was chosen to be 5:4, so that the ratio of a segment representing a degree in the east-west direction anywhere on the map to a segment representing a degree in the north-south direction is what it is on the globe at the latitude of Rhodes. As Ptolemy points out in 1.20, this means that the east-west spacing of places situated north or south of the parallel of Rhodes is progressively contracted the further south of Rhodes they are and progressively expanded the further north they are. The distortion would have been more pronounced in Marinos' map than in the map envisioned by Strabo, because Marinos' *oikoumenē* reaches significantly closer to the north pole than Strabo's does, and also extends to the equator and beyond, whereas Strabo assumed that the *oikoumenē* came to an end well north of the equator.

Ptolemy concedes this much to Marinos' choice of mapping, that if one imagines one's eye placed so that "the line of sight [is] directed at the middle of the northern quadrant of the sphere, in which most of the *oikoumenē* is mapped," and if the sphere is then revolved around its axis, each meridian in turn does appear as a straight line "when its plane falls through the apex of the sight." Hence, as a composite of a series of views of the sphere, the use of straight lines for meridians can be justified. But he also observes that to such an eye looking at the sphere, "the parallels… clearly give an appearance of circular segments bulging to the south," and a given pair of meridians "always cut off similar but unequal arcs on the parallels of different sizes, and always greater [arcs] on those nearer the equator."[49] Marinos' choice of projection lacks these properties.

[49]Not all of these statements are to be taken as a literal description of what is in fact seen, since in reality the parallels are seen as elliptical segments, not circular, and the portions of the arcs of different parallels between two meridians are not similar.

Ptolemy's First Map

After some further discussion, Ptolemy introduces the layout for his first map, in which he follows each statement about the geometry of the configuration with remarks about its effect (1.21):

> *First geometric criterion:* "It would be well to keep the lines representing the meridians straight, but [to have] those that represent the parallels as circular segments described about one and the same center, from which (imagined as the north pole) one will have to draw the meridian lines."
>
> *Its effect:* "Above all, the semblance of the spherical surface will be retained… with the meridians still remaining untilted with respect to the parallels [i.e., perpendicular to them] and still intersecting at that common pole."

> *Second geometric criterion:* "Since it is impossible to preserve for all the parallels their proportionality on the sphere, it would be adequate to keep this [proportionality] for the parallel through Thulē and the equator."
>
> *Its effect:* "The sides that enclose our [*oikoumenē*'s] latitudinal dimension [i.e., the bounding circular arcs representing the parallels of Thulē and the equator] will be in proper proportion to their true magnitudes."

> *Third geometric criterion:* "Divide [the parallel] that is to be drawn through Rhodes… in proportion to the meridian, that is in the approximate ratio of similar arcs of 5:4."
>
> *Its effect:* "The more familiar longitudinal dimension of the *oikoumenē* is in proper proportion to the latitudinal dimension."

The map that he produces has the following features:

1. The parallel bounding the *oikoumenē* on the north (the parallel through Thulē) is correctly represented relative to the size of the equator, i.e., the relative sizes of semicircles of the greatest and smallest parallels are correctly portrayed.
2. The longitudinal extent of the *oikoumenē* along the parallel of Rhodes relative to its latitudinal extent along the central meridian is faithfully represented on the map.
3. Each unit of distance along the straight lines representing the meridians between the parallels through anti-Meroē and Thulē faithfully rep-

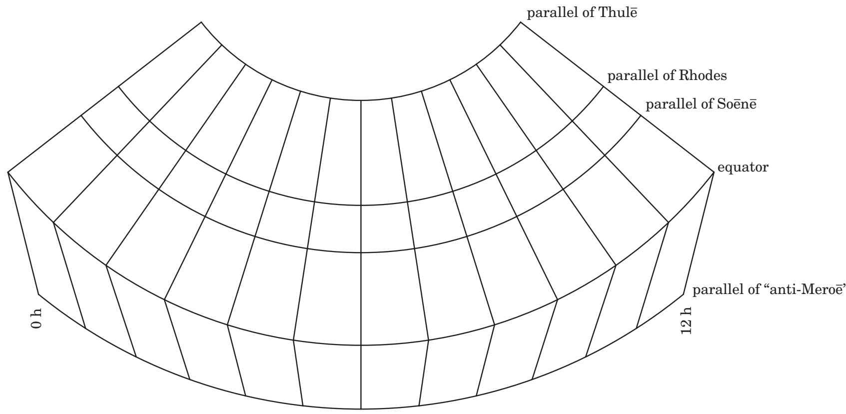

FIG. 11. Graticule of Ptolemy's first projection

resents one degree of arc on the corresponding meridians on the globe. This is sometimes phrased as "distances are preserved on all radii."[50]

In modern nomenclature, this is a version of the *simple conical* projection (conical projections in general are those in which parallels are represented by concentric circles, and meridians by straight lines intersecting at a single point). This projection has the property that east-west distances are portrayed in correct proportionality to north-south distances only along the selected parallel through Rhodes, and are progressively exaggerated the further north or south one goes from this parallel. In Ptolemy's view, the distortion becomes intolerable for parallels south of the equator, because from this point on the actual parallels on the globe diminish in circumference, while the arcs representing them on the map continue to increase in length.

To compensate for this unwanted effect, Ptolemy introduces an ad hoc modification of his graticule. The arc representing the southernmost parallel to be included in the map is shortened to make it equal in arc length to the arc standing for the parallel that is the same distance north of the equator, and east-west distances along the two parallels are therefore in correct proportion to each other (though not to the meridians). The graticule is completed by drawing the parts of the meridians between the equator and the southernmost parallel as straight lines joining equal longitudes on the corresponding arcs (Fig. 11). The part of the map south of the equator is thus in a *pseudoconical* projection. This adjustment introduces a practical difficulty for anyone drawing the map, since one can no longer use a swinging ruler pegged to the common inter-

[50] E.g., Neugebauer 1975a, 2:881. Note that if this proportionality of distances is continued beyond the upper limit of Ptolemy's map, the north pole will be represented not by point *H*, where the meridians all converge, but by an arc with radius 7 units from *H*.

section of the meridians to locate points south of the equator. It also compromises the mathematical consistency of the projection, but to castigate this as a fault is to impute to Ptolemy a concept of map projection that was not his own.[51]

Ptolemy's Second Map

In contrast to the first map, where the eye is thought of as passing over each meridian in turn, in the second map the eye and the globe remain fixed relative to each other. Ptolemy attempts to produce the impression of the meridians and parallels as they would be seen when the axis of the visual cone joins the eye to the center of the sphere and passes through the intersection of the central meridian and the central parallel of the *oikoumenē*,[52] the eye being sufficiently far away from the globe so that for all practical purposes it sees a hemisphere. Such an eye will perceive two semicircles of great circles as straight lines. One of these is the visible half of the central meridian, and the other is a great circle passing through the two poles of the central meridian and the city of Soēnē (chosen because it lies exactly on the Summer Tropic circle). On the other hand, the same eye will view (1) the other meridian circles as a series of arcs equally balanced on either side of the central meridian, like right and left parentheses but increasingly curved the farther they are from the central meridian, and (2) the visible portions of the parallel circles as a series of concentric circular arcs.

Ptolemy also wishes to do this in such a way that (1) the lengths of the arcs of the parallel circles represented have the correct ratio to each other, not just for the equator and the parallel through Thulē, as in his first map, but also "as very nearly as possible for the other" parallels, and that (2) his map preserves the ratio "of the total latitudinal dimension to the total longitudinal dimension… not only for the parallel drawn through Rhodes…, but [at least] roughly for absolutely all [the parallels]" (1.24).

To accomplish this, Ptolemy imagines the viewing eye as seeing the visible hemisphere as a circle, bisected by two perpendicular diameters, whose length he arbitrarily sets at 180 units (representing linearly the 180° of the semicircle in the direction of the eye). In terms of these units he establishes where the equator crosses the central meridian at 23⅚ units below the center (because the eye is supposed to be above Soēnē, which is at latitude 23⅚° north), and then where the center of the circle representing the equator will be. Since this will also be the center of the other parallel circles, he is now able to draw the circles on which the following parallels lie: that of "anti-Meroē" (marking the southern limit of the *oikoumenē*), that of Soēnē, and that of Thulē (at the north-

[51]Berggren 1991, 134–138.

[52]His central meridian cuts through the Persian Gulf, passes slightly to the west of Persepolis, and then heads northward through the Caspian Sea and Skythia. The central parallel of latitude is the parallel of Soēnē, in Lower Egypt.

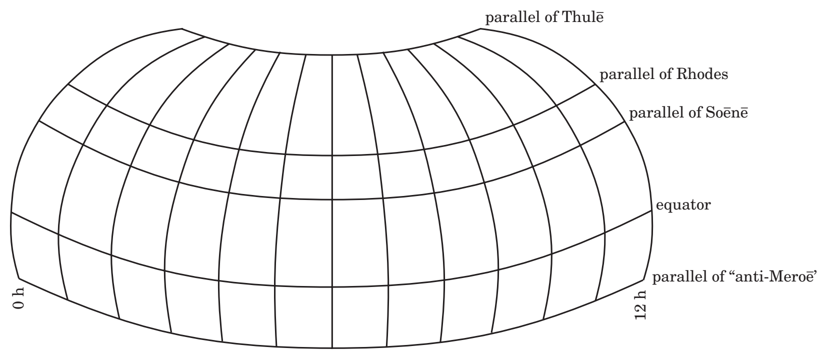

FIG. 12. Graticule of Ptolemy's second projection

ern limit). Taking arcs of five degrees to be equal to their chords, Ptolemy now marks off on each of these three parallels lengths corresponding to intervals of five degrees, and then joins triples of corresponding points with circular arcs to represent the meridians.

Figure 12 shows the resulting graticule. This is again a pseudoconical projection, since the parallels are drawn as concentric circular arcs, but the meridians are drawn as curves rather than as converging straight lines. Since a circular arc can be drawn through any three noncollinear points but not through any four, Ptolemy cannot keep distances measured along more than three of the parallels in exact proportionality with distances along the central meridian; and distances along the other meridians are distorted as a consequence of their curvature. If Ptolemy had made all the parallels proportionate in length to their actual lengths on the globe, and allowed the meridians to be drawn freely as the curves joining corresponding points on all parallels, he would have obtained the *Bonne* projection, which incidentally has the property of preserving areas of arbitrary regions on the globe.[53] Ptolemy, of course, would not have known this, and there is no suggestion that he, or any other ancient writer, had thought of preservation of areas as a desideratum in a map projection.

The Map in the Picture of the Ringed Globe

In 7.6, Ptolemy sets out a long geometrical construction of an image of the terrestrial globe surrounded by rings representing the principal circles of the celestial sphere. The construction consists of two distinct parts: determining points

[53]Printed editions of the *Geography* of the late fifteenth century gave either Ptolemy's first projection (e.g., the Rome edition of 1490, reproduced in Nordenskiöld 1889, plate I) or the second (e.g., the Ulm edition of 1482). Bernardus Sylvanus (1511) and Johannes Werner (1514) were the first to generalize Ptolemy's second projection along the lines described above; see Neugebauer 1975a, 2:885–888.

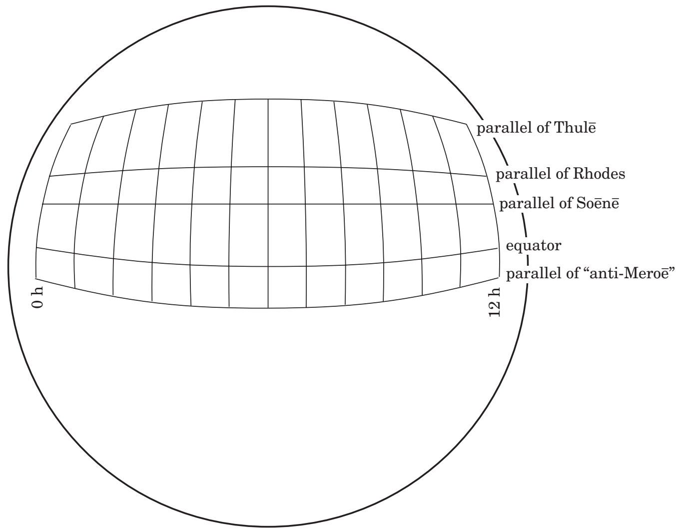

FIG. 13. Graticule of the projection in Ptolemy's picture of the ringed globe

through which the curves representing the various rings are to be drawn, and establishing a graticule for the map of the *oikoumenē* that is supposed to be visible between the rings. Ptolemy treats the two problems quite differently. The rings are drawn according to true linear perspective; that is, one imagines linear rays radiating from a point in space (representing the eye) through several points on each ring and onto a vertical plane, and the rings are drawn on that plane as ellipses passing through the projected points. This is in fact the unique example of a construction according to linear perspective surviving from antiquity. Ptolemy carries out the projection by treating the drawing plane first as the vertical plane containing the eye and the center of the globe, and thereafter as the plane of projection, which is at right angles to the former plane, so that the final drawing is made on the same plane as the geometrical construction of the perspective projection. This device, which eliminates the need for transferring measurements from one diagram to another, is reminiscent of the *analemma* constructions of sundial theory.[54]

Ptolemy's method for constructing the parallels of latitude portrayed on the terrestrial globe makes use of projected rays from the point representing the eye in a manner that superficially resembles the linear perspective used for the rings, but in fact the procedure has nothing to do with linear optics, and merely serves to generate a series of circular arcs that have their concavities facing a straight central parallel, thus qualitatively imitating the appearance of the actual parallel circles seen from an eyepoint in the plane of the chosen central parallel. The resulting projection (Fig. 13) resembles Ptolemy's second projec-

[54]Neugebauer 1975a, 2:839–856.

tion in using circular arcs to represent all meridians except the central one, which is a straight line, and in treating distances along this central meridian as proportional to the true distances on the globe; but the parallels are now laid out according to a plan analogous to that of the meridians instead of being drawn as concentric circular arcs. Again as in the second projection, proportionalities of distances are preserved along three parallels, along the top, bottom, and center of the map.

One may think of this third projection as a modification of Marinos' cylindrical projection, such that only one central meridian and one central parallel are drawn as straight lines in correct proportionality of distances, while the remaining parallels and meridians are drawn as circular arcs with curvature increasing as one goes further from the center of the map, to imitate the perspective appearance of the globe.

The Regional Maps

Having dismissed Marinos' cylindrical projection as unsuitable for a map of the entire *oikoumenē*, Ptolemy reintroduces it for the twenty-six regional maps into which he partitions the *oikoumenē* in Book 8. Each region is to be drawn in a graticule employing orthogonal straight lines for all meridians and parallels, and such that distances are represented in correct proportion along all meridians and along the central parallel for the region in question. The ratio of the lengths of one degree along the central parallel and along the meridians is stated in the caption to each regional map. Ptolemy left it to the cartographer, however, to find out just which meridians and parallels bound each region by finding the maximum and minimum longitudes and latitudes in the lists of coordinates. Someone after Ptolemy extracted these numbers, and listed them in a supplementary chapter that appears at the end of some manuscripts of the *Geography* (8.30 in Nobbe's edition).

The Coordinate Lists

Once the cartographer has constructed an appropriate graticule for the map of the world or one of the twenty-eight regions, the next task is to draw the map using the coordinate lists that make up the bulk of the *Geography* (Books 2.2–7.4). For this purpose, Ptolemy divided the *oikoumenē* into about eighty districts, which are grouped broadly into three continents (Europe, Libyē, and Asia), and within each continent are ordered loosely from west to east and from north to south. The chapters into which this part of the *Geography* is divided correspond to these districts.[55]

[55]It is not always obvious whether Ptolemy considers certain groupings of districts to belong together or not, so that the chapter divisions and the total number of districts are not definitely fixed.

Ptolemy refers to the districts in 2.1 as "provinces" and "satrapies," which would seem to identify them with the administrative divisions of the Roman and Parthian Empires, respectively. It does appear that Ptolemy (or Marinos) intended the districts contained by the Roman Empire to follow at least approximately the official borders of the provinces. On the other hand, Ptolemy's partition of Asia beyond the Roman frontier reflects the division of the Persian Empire into satrapies that was in effect in the time of Alexander the Great, and that had become part of the traditional apparatus of Greek geography. Ptolemy's map in fact makes little attempt to represent political geography, so that one cannot even tell from his map which districts belonged to the Roman Empire.

The map is composed of three kinds of object: one-dimensional (curvilinear) objects such as coastlines, the longer rivers, and some mountain ranges, which are to be drawn by connecting two or more points; pointlike objects such as cities, small islands, mountains, and the mouths of minor rivers; and peoples inhabiting small districts, who are located only in terms of the cities inside each district. Surprisingly, given that *itineraria* probably provided Marinos and Ptolemy with a good part of their geographical data, roads do not appear on the map. Each chapter begins with the definition of the outline of the province or satrapy, consisting of coastlines and borders (which sometimes coincide with rivers or mountain ranges). Borders that have already been described in a preceding chapter for the adjacent province are not repeated, and one generally has to refer back to the earlier chapter to find the last point from which drawing is to be continued. Cities and other features that lie on coasts are listed as part of the definition of the coastline. The coordinates defining the course of longer rivers are usually inserted as a digression at the point when the drawing of the coast has reached the river's mouth; for example, in 2.7 Ptolemy interrupts the description of the coast of Gallia Aquitanica when he has come to the mouth of the Garuna in order to insert the coordinates of two inland points that determine its course. The bends of some of the more complex rivers, such as the Nile, can only be drawn by inference after one has inscribed all the cities that are stated to be on one side or the other, a rare violation of Ptolemy's usual practice of giving specific longitudes and latitudes for all the cartographically significant points. Since the coordinates are specified only to the twelfth part of a degree, the resolution of the map is incapable of displaying distances smaller than that, so that, for example, the sizes and placements of offshore islands are not to scale.