Foreword

Constrained optimization is quite well established as an area of research, and there exist several powerful techniques that address general problems in that area. In this book a special class of constraints is considered, called geometric constraints, which express that the solution of the optimization problem lies on a manifold. This is a recent area of research that provides powerful alternatives to the more general constrained optimization methods. Classical constrained optimization techniques work in an embedded space that can be of a much larger dimension than that of the manifold. Optimization algorithms that work on the manifold have therefore a lower complexity and quite often also have better numerical properties (see, e.g., the numerical integration schemes that preserve invariants such as energy). The authors refer to this as unconstrained optimization in a constrained search space.

The idea that one can describe difference or differential equations whose solution lies on a manifold originated in the work of Brockett, Flaschka, and Rutishauser. They described, for example, isospectral flows that yield time-varying matrices which are all similar to each other and eventually converge to diagonal matrices of ordered eigenvalues. These ideas did not get as much attention in the numerical linear algebra community as in the area of dynamical systems because the resulting difference and differential equations did not lead immediately to efficient algorithmic implementations.

An important book synthesizing several of these ideas is *Optimization and Dynamical Systems* (Springer, 1994), by Helmke and Moore, which focuses on dynamical systems related to gradient flows that converge exponentially to a stationary point that is the solution of some optimization problem. The corresponding discrete-time version of this algorithm would then have linear convergence, which seldom compares favorably with state-of-the-art eigenvalue solvers.

The formulation of higher-order optimization methods on manifolds grew out of these ideas. Some of the people that applied these techniques to basic linear algebra problems include Absil, Arias, Chu, Dehaene, Edelman, Eldén, Gallivan, Helmke, Hüper, Lippert, Mahony, Manton, Moore, Sepulchre, Smith, and Van Dooren. It is interesting to see, on the other hand, that several basic ideas in this area were also proposed by Luenberger and Gabay in the optimization literature in the early 1980s, and this without any use of dynamical systems.

In the present book the authors focus on higher-order methods and include Newton-type algorithms for optimization on manifolds. This requires

a lot more machinery, which cannot currently be found in textbooks. The main focus of this book is on optimization problems related to invariant subspaces of matrices, but this is sufficiently general to encompass well the two main aspects of optimization on manifolds: the conceptual algorithm and its convergence analysis based on ideas of differential geometry, and the efficient numerical implementation using state-of-the-art numerical linear algebra techniques.

The book is quite deep in the presentation of the machinery of differential geometry needed to develop higher-order optimization techniques, but it nevertheless succeeds in explaining complicated concepts with simple ideas. These ideas are then used to develop Newton-type methods as well as other superlinear methods such as trust-region methods and inexact and quasi-Newton methods, which precisely put more emphasis on the efficient numerical implementation of the conceptual algorithms.

This is a research monograph in a field that is quickly gaining momentum. The techniques are also being applied to areas of engineering and robotics, as indicated in the book, and it sheds new light on methods such as the Jacobi-Davidson method, which originally came from computational chemistry. The book makes a lot of interesting connections and can be expected to generate several new results in the future.

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